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ACYCLIC RESOLUTIONS FOR ARBITRARY GROUPS

BY

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ABSTRACT

We prove that for every abelian group G and every compactum X with $\dim_G X \leq n \geq 2$ there is a G-acyclic resolution $r: Z \longrightarrow X$ from a compactum Z with $\dim_G Z \leq n$ and $\dim Z \leq n+1$ onto X.

1. Introduction

All spaces are assumed to be separable metrizable. A compactum is a metrizable compact space.

Let G be an abelian group. A space X has the cohomological dimension $\dim_G X \leq n$ if $\check{H}^{n+1}(X, A; G) = 0$ for every closed subset A of X. The case $G = \mathbb{Z}$ is an important special case of cohomological dimension. It was known long ago that $\dim X = \dim_{\mathbb{Z}} X$ if X is finite dimensional. Solving an outstanding problem in cohomological dimension theory Dranishnikov constructed in 1987 an infinite dimensional compactum of $\dim_{\mathbb{Z}} = 3$. A few years earlier a deep relation between $\dim_{\mathbb{Z}}$ and dim was established by the Edwards cell-like resolution theorem [4, 9] saying that a compactum of $\dim_{\mathbb{Z}} \leq n$ can be obtained as the image of a cell-like map defined on a compactum of dim $\leq n$. A compactum X is cell-like if any map $f: X \longrightarrow K$ from X to a CW-complex K is null homotopic. A map is cell-like if its fibers are cell-like. The reduced Čech cohomology groups of a cell-like compactum are trivial with respect to any group G.

Acyclic resolutions originated in the Edwards cell-like resolution. A compactum X is G-acyclic if $\tilde{\check{H}}^*(X; G) = 0$ and a map is G-acyclic if its fibers are

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G-acyclic. Thus a cell-like map is G-acyclic with respect to any abelian group G. By the Vietoris–Begle theorem a G-acyclic map cannot raise the cohomological dimension $\dim_G$. Dranishnikov proved the following important

THEOREM 1.1 ([2, 3]): *Let X be a compactum with $\dim_{\mathbb{Q}} X \leq n$, $n \geq 2$. Then there are a compactum Z with $\dim_{\mathbb{Q}} Z \leq n$ and $\dim Z \leq n+1$ and a $\mathbb{Q}$-acyclic map $r\colon Z \longrightarrow X$ from Z onto X.*

It has been widely conjectured that Theorem 1.1 holds for any abelian group G. Substantial progress in solving this conjecture was made by Koyama and Yokoi [6] who proved it for a large class of groups including $\mathbb{Q}$ and very recently by Rubin and Schapiro [8] who settled the case $G = \mathbb{Z}_{p^\infty}$.

The purpose of this note is to finally answer this conjecture affirmatively by proving

THEOREM 1.2: *Let G be an abelian group and let X be a compactum with $\dim_G X \leq n$, $n \geq 2$. Then there are a compactum Z with $\dim_G Z \leq n$ and $\dim Z \leq n+1$ and a G-acyclic map $r\colon Z \longrightarrow X$ from Z onto X.*

It is known that the dimension $n+1$ of Z in Theorem 1.2 is best possible [6]. However, it is unknown if the dimension of Z in Theorem 1.1 can be reduced to n (see [5] for related results). In this connection let us also mention the following interesting result of Dranishnikov.

THEOREM 1.3 ([1]): *Let X be a compactum with $\dim_{\mathbb{Z}_p} X \leq n$. Then there are a compactum Z with $\dim Z \leq n$ and a $\mathbb{Z}_p$-acyclic map $r\colon Z \longrightarrow X$ from Z onto X.*

Our proof of Theorem 1.2 essentially uses Dranishnikov's idea of constructing a $\mathbb{Q}$-acyclic resolution presented in [3] and involves some methods of [7]. The proof is self-contained and does not rely on previous results concerning acyclic resolutions. The paper [3] is an excellent source of basic information on cohomological dimension theory.

2. Preliminaries

All groups below are abelian and functions between groups are homomorphisms. $\mathcal{P}$ stands for the set of primes. For a non-empty subset $\mathcal{A}$ of $\mathcal{P}$ let $S(\mathcal{A}) = \{p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k} : p_i \in \mathcal{A}, n_i \geq 0\}$ be the set of positive integers with prime factors from $\mathcal{A}$ and for the empty set define $S(\emptyset) = \{1\}$. Let G be a group and $g \in G$. We say that g is $\mathcal{A}$-torsion if there is $n \in S(\mathcal{A})$ such that $ng = 0$ and g is $\mathcal{A}$-divisible if for every $n \in S(\mathcal{A})$ there is $h \in G$ such that $nh = g$. $\operatorname{Tor}_{\mathcal{A}} G$ is the subgroup of the $\mathcal{A}$-torsion elements of G. G is $\mathcal{A}$-torsion if $G = \operatorname{Tor}_{\mathcal{A}} G$, G is $\mathcal{A}$-torsion free if $\operatorname{Tor}_{\mathcal{A}} G = 0$ and G is $\mathcal{A}$-divisible if every element of G is $\mathcal{A}$-divisible.

PROPOSITION 2.1:

(i) *If G is $\mathcal{A}$-torsion then G is $(\mathcal{P} \setminus \mathcal{A})$-divisible and $(\mathcal{P} \setminus \mathcal{A})$-torsion free.*

(ii) *A factor group of an $\mathcal{A}$-divisible group is $\mathcal{A}$-divisible and a factor group of an $\mathcal{A}$-torsion group is $\mathcal{A}$-torsion.*

(iii) *The direct sum of $\mathcal{A}$-divisible groups is $\mathcal{A}$-divisible and the direct sum of $\mathcal{A}$-torsion groups is $\mathcal{A}$-torsion.*

Let $f\colon G \longrightarrow H$ be a homomorphism of groups G and H and let H be $\mathcal{B}$-torsion. Then $G/\operatorname{Tor}_{\mathcal{B}} G$ is

(iv) *$\mathcal{A}$-divisible if $\ker f$ is $\mathcal{A}$-divisible and $\mathcal{B} \subset \mathcal{A}$;*

(v) *$\mathcal{A}$-torsion if $\ker f$ is $\mathcal{A}$-torsion and $\mathcal{B} \cap \mathcal{A} = \emptyset$;*

(vi) *$\mathcal{A}$-torsion and $\mathcal{A}$-divisible if $\ker f$ is $\mathcal{A}$-torsion and $\mathcal{A}$-divisible and $\mathcal{B} \cap \mathcal{A} = \emptyset$.*

Proof: The proof of (i), (ii), (iii) is obvious.

Let $\phi\colon G \longrightarrow G/\operatorname{Tor}_{\mathcal{B}} G$ be the projection and $\phi(x) = y$. Then there is $n \in S(\mathcal{B})$ such that $nf(x) = f(nx) = 0$ and hence $nx \in \ker f$.

(iv) Let $m \in S(\mathcal{A})$. Since $\mathcal{B} \subset \mathcal{A}$, $nm \in S(\mathcal{A})$. Then there is $z \in \ker f$ such that $nmz = nx$. Hence $n(mz - x) = 0$ and therefore $\phi(mz - x) = 0$. Thus $m\phi(z) = \phi(x) = y$ and $G/\operatorname{Tor}_{\mathcal{B}} G$ is $\mathcal{A}$-divisible.

(v) By (i), $\ker f$ is $(\mathcal{P} \setminus \mathcal{A})$-divisible and therefore there is $z \in \ker f$ such that $nz = nx$. Then $n(z - x) = 0$ and there is $m \in S(\mathcal{A})$ such that $mz = 0$. Hence $\phi(z) = \phi(x) = y$ and $my = \phi(mz) = 0$ and (v) follows.

(vi) By (v), $G/\operatorname{Tor}_{\mathcal{B}} G$ is $\mathcal{A}$-torsion. By (i), $\ker f$ is $(\mathcal{P} \setminus \mathcal{A})$-divisible and, since $\ker f$ is $\mathcal{A}$-divisible, $\ker f$ is $\mathcal{P}$-divisible. Then by (iv), $G/\operatorname{Tor}_{\mathcal{B}} G$ is $\mathcal{A}$-divisible. ∎

The notation $\mathrm{e} - \dim X \le Y$ is used to indicate the property that every map $f\colon A \longrightarrow Y$ of a closed subset A of X into Y extends over X. It is known that $\dim_G X \le n$ if and only if $\mathrm{e} - \dim X \le K(G, n)$ where $K(G, n)$ is the Eilenberg–Mac Lane complex of type (G, n). A map between CW-complexes is combinatorial if the preimage of every subcomplex of the range is a subcomplex of the domain.

Let M be a simplicial complex and let $M^{[n]}$ be the n-skeleton of M (=the union of all simplexes of M of dim $\le n$). By a resolution $EW(M, n)$ of M we mean a CW-complex $EW(M, n)$ and a combinatorial map $\omega\colon EW(M, n) \longrightarrow M$ such that ω is 1-to-1 over $M^{[n]}$. The resolution is said to be suitable for a map

$f\colon M^{[n]} \longrightarrow Y$ if the map $f \circ \omega|_{\omega^{-1}(M^{[n]})}$ extends to a map from $EW(M,n)$ to Y. The resolution is said to be suitable for a compactum X if for every simplex Δ of M, $\mathrm{e}-\dim X \leq \omega^{-1}(\Delta)$. Note that if $\omega\colon EW(M,n) \longrightarrow M$ is a resolution suitable for X, then for every map $\phi\colon X \longrightarrow M$ there is a map $\psi\colon X \longrightarrow EW(M,n)$ such that for every simplex Δ of M, $(\omega \circ \psi)(\phi^{-1}(\Delta)) \subset \Delta$. We will call ψ a combinatorial lifting of ϕ.

Following [7] we will construct a resolution of an $(n+1)$-dimensional simplicial complex M which is suitable for X with $\dim_G X \leq n$ and a map $f\colon M^{[n]} \longrightarrow K(G,n)$. In the sequel we will refer to this resolution as the standard resolution for f. Fix a CW-structure on $K(G,n)$ and assume that f is cellular. We will obtain a CW-complex $EW(M,n)$ from $M^{[n]}$ by attaching the mapping cylinder of $f|_{\partial\Delta}$ to $\partial\Delta$ for every $(n+1)$-simplex Δ of M. Let $\omega\colon EW(M,n) \longrightarrow M$ be the projection sending each mapping cylinder to the corresponding $(n+1)$-simplex Δ such that ω is the identity map on $\partial\Delta$, the $K(G,n)$-part of the cylinder is sent to the barycenter of Δ and ω is 1-to-1 on the rest of the cylinder. Clearly $f|_{\partial\Delta}$ extends over its mapping cylinder and therefore $f \circ \omega|_{\omega^{-1}(M^{[n]})}$ extends over $EW(M,n)$. For each simplex Δ of M, $\omega^{-1}(\Delta)$ is either contractible or homotopy equivalent to $K(G,n)$. Define a CW-structure on $EW(M,n)$ turning ω into a combinatorial map. Thus we get that the standard resolution is indeed a resolution suitable for both X and f. Note that from the construction of the standard resolution it follows that for every subcomplex T of M, $\omega^{-1}(T)$ is the standard resolution of T for $f|_{T^{[n]}}$ and $\omega^{-1}(T)$ is $(n-1)$-connected if T is $(n-1)$-connected.

Proposition 2.2: *Let M be an $(n+1)$-dimensional finite simplicial complex and let $\omega\colon EW(M,n) \longrightarrow M$ be the standard resolution for $f\colon M^{[n]} \longrightarrow K(G,n)$, $n \geq 2$. Then for $\omega_*\colon H_n(EW(M,n)) \longrightarrow H_n(M)$, $\ker\omega_*$ is a factor group of the direct sum $\bigoplus G$ of finitely many G.*

Proof: Inside each $(n+1)$-simplex of M cut a small closed ball around the barycenter and not touching the boundary and split M into two subspaces $M = M_1 \cup M_2$ where $M_1=$ the closure of the complement to the union of the balls and M_2=the union of the balls. Then ω is 1-to-1 over M_1, $H_{n-1}(M_1 \cap M_2) = 0$, $H_n(M_2) = 0$ and the preimage under ω of each ball is homotopy equivalent to $K(G,n)$ and hence $H_n(\omega^{-1}(M_2))$ is the direct sum $\bigoplus G$ of finitely many G. Consider the Mayer–Vietoris sequences for the pairs (M_1, M_2) and $(\omega^{-1}(M_1), \omega^{-1}(M_2))$, in which we identify M_1 and $M_1 \cap M_2$ with $\omega^{-1}M_1$ and $\omega^{-1}(M_1 \cap M_2)$, respectively.

From the Mayer–Vietoris sequences it follows that

$$j_*(H_n(\omega^{-1}(M_1) \oplus H_n(\omega^{-1}(M_2))) = H_n(\omega^{-1}(M_1 \cup M_2))$$

and $j_*(0 \oplus H_n(\omega^{-1}(M_2))) \subset \ker \omega_*$. Let us show that $j_*(0 \oplus H_n(\omega^{-1}(M_2))) \supset \ker \omega_*$. Let $j_*(a \oplus b) \in \ker \omega_*$. Then in the Mayer–Vietoris sequence for the pair (M_1, M_2), $j_*(a \oplus 0) = 0$ and therefore there is $c \in H_n(M_1 \cap M_2))$ such that $i_*(c) = a \oplus 0$. Then in the Mayer–Vietoris sequence for the pair $(\omega^{-1}(M_1), \omega^{-1}(M_2))$, $i_*(c) = a \oplus d$ and $j_*(a \oplus d) = 0$. Thus $j_*(a \oplus b) = j_*(0 \oplus (b-d))$ and therefore $j_*(0 \oplus H_n(\omega^{-1}(M_2))) = \ker \omega_*$. Recall that $H_n(\omega^{-1}(M_2)) = \bigoplus G$ and the proposition follows. ∎

PROPOSITION 2.3: *Let $M = M_1 \cup M_2$ be a CW-complex with subcomplexes M_1 and M_2 such that M_1, M_2 and $M_1 \cap M_2$ are $(n-1)$-connected, $n \geq 2$. Then M is $(n-1)$-connected and*

(i) *$H_n(M)$ is $\mathcal{A}$-divisible if $H_n(M_1)$ and $H_n(M_2)$ are $\mathcal{A}$-divisible;*
(ii) *$H_n(M)$ is $\mathcal{A}$-torsion if $H_n(M_1)$ and $H_n(M_2)$ are $\mathcal{A}$-torsion.*

Proof: The connectedness of M follows from van Kampen and Hurewicz's theorems and the Mayer–Vietoris sequence. (i) and (ii) follow from the Mayer–Vietoris sequence and (ii) and (iii) of Proposition 2.1. ∎

Let X be a compactum and let $\sigma(G)$ be the Bockstein basis of a group G. By Bockstein's theory $\dim_G X \leq n$ if and only if $\dim_E X \leq n$ for every $E \in \sigma(G)$. Denote:

$\mathcal{T}(G) = \{p \in \mathcal{P} : \mathbb{Z}_p \in \sigma(G)\}$;
$\mathcal{T}_\infty(G) = \{p \in \mathcal{P} : \mathbb{Z}_{p^\infty} \in \sigma(G)\}$;
$\mathcal{D}(G) = \mathcal{P}$ if $\mathbb{Q} \in \sigma(G)$ and $\mathcal{D}(G) = \mathcal{P} \setminus \{p \in \mathcal{P} : \mathbb{Z}_{(p)} \in \sigma(G)\}$ otherwise;
$\mathcal{F}(G) = \mathcal{D}(G) \setminus (\mathcal{T}(G) \cup \mathcal{T}_\infty(G))$.

Note that $\mathcal{T}(G)$, $\mathcal{T}_\infty(G)$ and $\mathcal{F}(G)$ are disjoint and G is $\mathcal{F}(G)$-torsion free.

PROPOSITION 2.4: *Let X be a compactum and let G be an abelian group such that $G/\operatorname{Tor} G \neq 0$ and $\dim_G X \leq n$. Then $\dim_E X \leq n$ if E is $\mathcal{D}(G)$-divisible and $\mathcal{F}(G)$-torsion free.*

Proof: The proof is based on Bockstein's theorem and inequalities.

If $\mathbb{Z}_p \in \sigma(E)$ then $\operatorname{Tor}_p E$ is not divisible by p and hence E is not divisible by p. Thus $p \in \mathcal{P} \setminus \mathcal{D}(G)$ and therefore $\mathbb{Z}_{(p)} \in \sigma(G)$ and $\dim_{\mathbb{Z}_p} X \leq \dim_{\mathbb{Z}_{(p)}} X \leq n$.

If $\mathbb{Z}_{p^\infty} \in \sigma(E)$ then p is not in $\mathcal{F}(G)$. Then either $p \in \mathcal{P} \setminus \mathcal{D}(G)$ and $\dim_{\mathbb{Z}_{p^\infty}} X \leq \dim_{\mathbb{Z}_{(p)}} X \leq n$ or $p \in \mathcal{D}(G) \setminus \mathcal{F}(G)$ and then either $p \in \mathcal{T}(G)$ and $\dim_{\mathbb{Z}_{p^\infty}} X \leq \dim_{\mathbb{Z}_p} X \leq n$ or $p \in \mathcal{T}_\infty(G)$ and $\dim_{\mathbb{Z}_{p^\infty}} X \leq n$.

If $\mathbb{Z}_{(p)} \in \sigma(E)$ then $E/\mathrm{Tor}\, E$ is not divisible by p and hence E is not divisible by p. Thus $p \in \mathcal{P} \setminus \mathcal{D}(G)$ and therefore $\dim_{\mathbb{Z}_{(p)}} X \leq n$.

If $\mathbb{Q} \in \sigma(E)$ then consider the following cases:

(i) $\mathcal{D}(G) = \mathcal{P}$. Then since $G/\mathrm{Tor}\, G \neq 0$, $\mathbb{Q} \in \sigma(G)$ and therefore $\dim_{\mathbb{Q}} X \leq n$ (this is the only place where we use that $G/\mathrm{Tor}\, G \neq 0$).

(ii) There is $p \in \mathcal{P} \setminus \mathcal{D}(G)$. Then $\dim_{\mathbb{Q}} X \leq \dim_{\mathbb{Z}_{(p)}} X \leq n$. ∎

3. Proof of Theorem 1.2

Represent X as the inverse limit $X = \varprojlim(K_i, h_i)$ of simplicial complexes K_i with combinatorial bonding maps $h_{i+1}\colon K_{i+1} \longrightarrow K_i$ onto and the projections $p_i\colon X \longrightarrow K_i$ such that for every simplex Δ of K_i, $\mathrm{diam}(p_i^{-1}(\Delta)) \leq 1/i$. Following A. Dranishnikov [3] we construct by induction CW-complexes L_i and maps $g_{i+1}\colon L_{i+1} \longrightarrow L_i$, $\alpha_i\colon L_i \longrightarrow K_i$ such that:

(a) L_i is $(n+1)$-dimensional and obtained from $K_i^{[n+1]}$ by replacing some $(n+1)$-simplexes by $(n+1)$-cells attached to the boundary of the replaced simplexes by a map of degree $\in S(\mathcal{F}(G))$. Then α_i is a projection of L_i taking the new cells to the original ones such that α_i is 1-to-1 over $K_i^{[n]}$. We define a simplicial structure on L_i for which α_i is a combinatorial map and refer to this simplicial structure while constructing resolutions of L_i. Note that for $\mathcal{F}(G) = \emptyset$ we don't replace simplexes of $K_i^{[n+1]}$ at all.

(b) The maps h_i, g_i and α_i combinatorially commute. By this we mean that for every simplex Δ of K_i, $(\alpha_i \circ g_{i+1})((h_{i+1} \circ \alpha_{i+1})^{-1}(\Delta)) \subset \Delta$.

We will construct L_i in such a way that $Z = \varprojlim(L_i, g_i)$ will be of $\dim_G \leq n$ and Z will admit a G-acyclic map onto X.

Set $L_1 = K_1^{[n+1]}$ with $\alpha_1\colon L_1 \longrightarrow K_1$ the embedding and assume that the construction is completed for i. Let $E \in \sigma(G)$ and let $f\colon L_i^{[n]} \longrightarrow K(E, n)$ be a cellular map. Let $\omega_L\colon EW(L_i, n) \longrightarrow L_i$ be the standard resolution of L_i for f. We are going to construct from $EW(L_i, n)$ a resolution of K_i suitable for X. On the first step of the construction we will obtain from $EW(L_i, n)$ a resolution $\omega_{n+1}\colon EW(K_i^{[n+1]}, n) \longrightarrow K_i^{[n+1]}$ such that $EW(L_i, n)$ is a subcomplex of $EW(K_i^{[n+1]}, n)$ and ω_{n+1} extends $\alpha_i \circ \omega_L$. On the second step we will construct resolutions $\omega_j\colon EW(K_i^{[j]}, n) \longrightarrow K_i^{[j]}$, $n+2 \leq j \leq \dim K_i$ such that $EW(K_i^{[j]}, n)$ is a subcomplex of $EW(K_i^{[j+1]}, n)$ and ω_{j+1} extends ω_j for $n+1 \leq j < \dim K_i$. From the construction below it will follow that for every $n+1 \leq j \leq \dim K_i$ the n-skeletons of $EW(L_i, n)$ and $EW(K_i^{[j]}, n)$ coincide. Noe that $(\alpha_i \circ \omega_L)^{-1}(T)$ is $(n-1)$-connected if T is an $(n-1)$-connected subcomplex of K_i. Then $\omega_j^{-1}(T)$

is also $(n-1)$-connected if T is an $(n-1)$-connected subcomplex of $K_i^{[j]}$. The construction is carried out as follows.

STEP 1: For every simplex Δ of K_i of $\dim = n+1$ consider separately the subcomplex $(\alpha_i \circ \omega_L)^{-1}(\Delta)$ of $EW(L_i, n)$. Enlarge $(\alpha_i \circ \omega_L)^{-1}(\Delta)$ by attaching cells of $\dim = n+1$ in order to kill $\mathrm{Tor}_{\mathcal{F}(G)} H_n((\alpha_i \circ \omega_L)^{-1}(\Delta))$ and attaching cells of $\dim > n+1$ in order to kill all homotopy groups of the enlarged subcomplex in $\dim > n$. Define $EW(K_i^{[n+1]}, n)$ as $EW(L_i, n)$ with all the cells attached for all $(n+1)$-dimensional simplexes Δ of K_i and let a map $\omega_{n+1}\colon EW(K_i^{[n+1]}, n) \longrightarrow K_i^{[n+1]}$ extend $\alpha_i \circ \omega_L$ by sending the interior points of the attached cells to the interior of the corresponding Δ.

STEP 2: Assume that a resolution $\omega_j\colon EW(K_i^{[j]}, n) \longrightarrow K_i^{[j]}$, $n+1 \le j < \dim K_i$ is constructed. For every simplex Δ of K_i of $\dim = j+1$ consider separately the subcomplex ${\omega_j}^{-1}(\partial\Delta)$ of $EW(K_i^{[j]}, n)$. Enlarge ${\omega_j}^{-1}(\partial\Delta)$ by attaching cells of $\dim = n+1$ in order to kill $\mathrm{Tor}_{\mathcal{F}(G)} H_n(\omega_j^{-1}(\partial\Delta))$ and attaching cells of $\dim > n+1$ in order to kill all homotopy groups of the enlarged subcomplex in $\dim > n$. Define $EW(K_i^{[j+1]}, n)$ as $EW(K_i^{[j]}, n)$ with all the cells attached for all $(j+1)$-simplexes Δ of K_i and let a map $\omega_{j+1}\colon EW(K_i^{[j+1]}, n) \longrightarrow K_i^{[j+1]}$ extend ω_j by sending the interior points of the attached cells to the interior of the corresponding Δ.

Finally, denote $EW(K_i, n) = EW(K_i^{[m]}, n)$ and $\omega = \omega_m\colon EW(K_i, n) \longrightarrow K_i$ where $m = \dim K_i$. Note that from the construction it follows that the n-skeleton of $EW(K_i, n)$ is contained in $EW(L_i, n)$.

Let us show that $EW(K_i, n)$ is suitable for X. First note that for every simplex Δ of $\dim \ge n+1$, $\omega^{-1}(\Delta)$ is homotopy equivalent to $K(H_n(\omega^{-1}(\Delta)), n)$.

In order to show that $\dim_{H_n(\omega^{-1}(\Delta))} X \le n$ we first consider Step 1 of the construction. Let Δ be an $(n+1)$-dimensional simplex of K_i. By $\omega_L|_{\dots}$ we will denote the map $\omega_L|_{(\alpha_i \circ \omega_L)^{-1}(\Delta)}\colon (\alpha_i \circ \omega_L)^{-1}(\Delta) \longrightarrow {\alpha_i}^{-1}(\Delta)$ with the range restricted to ${\alpha_i}^{-1}(\Delta)$. Let $(\omega_L|\dots)_*\colon H_n((\alpha_i \circ \omega_L)^{-1}(\Delta)) \to H_n(\alpha_i^{-1}(\Delta))$ be the induced homomorphism. Note that by (a), $H_n(\alpha_i^{-1}(\Delta))$ is $\mathcal{F}(G)$-torsion and $H_n(\omega^{-1}(\Delta)) = H_n((\alpha_i \circ \omega_L)^{-1}(\Delta)) / \mathrm{Tor}_{\mathcal{F}(G)} H_n((\alpha_i \circ \omega_L)^{-1}(\Delta))$. Consider the following cases.

CASE 1-1: $E = \mathbb{Z}_p$. By Proposition 2.2, $\ker(\omega_L|_{\dots})_*$ is p-torsion. Then since p is not in $\mathcal{F}(G)$, by Proposition 2.1, (v), $H_n(\omega^{-1}(\Delta))$ is p-torsion and by Bockstein's theorem $\dim_{H_n(\omega^{-1}(\Delta))} X \le \dim_{\mathbb{Z}_p} X \le n$.

CASE 1-2: $E = \mathbb{Z}_{p^\infty}$. By Proposition 2.2, $\ker(\omega_L|_{\dots})_*$ is p-torsion and p-divisible. Then since p is not in $\mathcal{F}(G)$, by Proposition 2.1, (vi), $H_n(\omega^{-1}(\Delta))$ is p-

torsion and p-divisible and by Bockstein's theorem $\dim_{H_n(\omega^{-1}(\Delta))} X \leq \dim_{\mathbb{Z}_{p^\infty}} X \leq n$.

CASE 1-3: $E = \mathbb{Z}_{(p)}$ or $E = \mathbb{Q}$. By Proposition 2.2, $\ker(\omega_L|_{\dots})_*$ is $\mathcal{D}(G)$-divisible. Then since $\mathcal{F}(G) \subset \mathcal{D}(G)$, by Proposition 2.1, (iv), $H_n(\omega^{-1}(\Delta))$ is $\mathcal{D}(G)$-divisible and since $H_n(\omega^{-1}(\Delta))$ is $\mathcal{F}(G)$-torsion free, by Proposition 2.4, $\dim_{H_n(\omega^{-1}(\Delta))} X \leq n$.

Now let us pass to Step 2 of the construction. We will show that the properties of the homology groups established above will be preserved for simplexes of higher dimensions. Let Δ be a $(j+1)$-dimensional simplex of K_i, $j \geq n+1$ and recall that $H_n(\omega^{-1}(\Delta)) = H_n(\omega^{-1}(\partial\Delta))/\operatorname{Tor}_{\mathcal{F}(G)} H_n(\omega^{-1}(\partial\Delta))$. Note that $\omega^{-1}(T)$ is $(n-1)$-connected if T is an $(n-1)$-connected subcomplex of K_i. We use this fact when we apply below Proposition 2.3 to show that while assembling $\omega^{-1}(\partial\Delta)$ from $\omega^{-1}(\Delta')$ for j-simplexes Δ' of Δ we preserve the corresponding properties of $\omega^{-1}(\Delta')$. Once again we consider separately the following cases.

CASE 2-1: $E = \mathbb{Z}_p$. If for every j-dimensional simplex Δ' of Δ, $H_n(\omega^{-1}(\Delta'))$ is p-torsion then, by Proposition 2.3, $H_n(\omega^{-1}(\partial\Delta))$ is p-torsion and hence $H_n(\omega^{-1}(\Delta))$ is p-torsion. Therefore $\dim_{H_n(\omega^{-1}(\Delta))} X \leq \dim_{\mathbb{Z}_p} X \leq n$.

CASE 2-2: $E = \mathbb{Z}_{p^\infty}$. If for every j-dimensional simplex Δ' of Δ, $H_n(\omega^{-1}(\Delta'))$ is p-torsion and p-divisible then, by Proposition 2.3, $H_n(\omega^{-1}(\partial\Delta))$ is p-torsion and p-divisible and hence $H_n(\omega^{-1}(\Delta))$ is p-torsion and p-divisible. Therefore $\dim_{H_n(\omega^{-1}(\Delta))} X \leq \dim_{\mathbb{Z}_{p^\infty}} X \leq n$.

CASE 2-3: $E = \mathbb{Z}_{(p)}$ or $E = \mathbb{Q}$. If for every j-dimensional simplex Δ' of Δ, $H_n(\omega^{-1}(\Delta'))$ is $\mathcal{D}(G)$-divisible then, by Proposition 2.3, $H_n(\omega^{-1}(\partial\Delta))$ is $\mathcal{D}(G)$-divisible. Then $H_n(\omega^{-1}(\Delta))$ is $\mathcal{D}(G)$-divisible and $\mathcal{F}(G)$-torsion free and, by Proposition 2.4, $\dim_{H_n(\omega^{-1}(\Delta))} X \leq n$.

Thus we have shown that $EW(K_i, n)$ is suitable for X. Now replacing K_{i+1} by a K_l with a sufficiently large l we may assume that there is a combinatorial lifting of h_{i+1} to $h'_{i+1}: K_{i+1} \longrightarrow EW(K_i, n)$. Replace h'_{i+1} by its cellular approximation preserving the property of h'_{i+1} of being a combinatorial lifting of h_{i+1}.

Consider the $(n+1)$-skeleton of K_{i+1} and let Δ_{i+1} be an $(n+1)$-dimensional simplex in K_{i+1}. Let Δ_i be the smallest simplex in K_i containing $h_{i+1}(\Delta_{i+1})$. Then $h'_{i+1}(\Delta_{i+1}) \subset \omega^{-1}(\Delta_i)$. Let τ: $(\alpha_i \circ \omega_L)^{-1}(\Delta_i) \longrightarrow \omega^{-1}(\Delta_i)$ be the inclusion. Note that from the construction it follows that for

$$\tau_*: H_n((\alpha_i \circ \omega_L)^{-1}(\Delta_i)) \longrightarrow H_n(\omega^{-1}(\Delta_i)),$$

$\ker \tau_*$ is $\mathcal{F}(G)$-torsion. Recall that the n-skeleton of $\omega^{-1}(\Delta_i)$ is contained in $(\alpha_i \circ \omega_L)^{-1}(\Delta_i)$ and consider $h'_{i+1}|_{\partial\Delta_{i+1}}$ as a map to $(\alpha_i \circ \omega_L)^{-1}(\Delta_i)$. Let a be the generator of $H_n(\partial\Delta_{i+1})$. Since $h'_{i+1}|_{\partial\Delta_{i+1}}$ extends over Δ_{i+1} as a map to $\omega^{-1}(\Delta_i)$ we have that $\tau_*((h'_{i+1}|_{\partial\Delta_{i+1}})_*(a)) = 0$. Hence $(h'_{i+1}|_{\partial\Delta_{i+1}})_*(a) \in \ker \tau_*$ and therefore there is $k \in S(\mathcal{F}(G))$ such that $k((h'_{i+1}|_{\partial\Delta_{i+1}})_*(a)) = 0$. Replace Δ_{i+1} by a cell C attached to the boundary of Δ_{i+1} by a map of degree k if $(h'_{i+1}|_{\partial\Delta_{i+1}})_*(a) \neq 0$ and set $C = \Delta_{i+1}$ if $(h'_{i+1}|_{\partial\Delta_{i+1}})_*(a) = 0$. Then $h'_{i+1}|_{\partial\Delta_{i+1}}$ can be extended over C as a map to $(\alpha_i \circ \omega_L)^{-1}(\Delta_i)$ and we will denote this extension by $g'_{i+1}|_C\colon C \longrightarrow (\alpha_i \circ \omega_L)^{-1}(\Delta_i)$. Thus replacing if needed $(n+1)$-simplexes of $K_{i+1}^{[n+1]}$ we construct from $K_{i+1}^{[n+1]}$ a CW-complex L_{i+1} and a map $g'_{i+1}\colon L_{i+1} \longrightarrow EW(L_i, n)$ which extends h'_{i+1} restricted to the n-skeleton of K_{i+1}. Now define $g_{i+1} = \omega_L \circ g'_{i+1}\colon L_{i+1} \longrightarrow L_i$ and finally define a simplicial structure on L_{i+1} for which α_{i+1} is a combinatorial map. It is easy to check that the properties (a) and (b) are satisfied. Since the triangulation of L_{i+1} can be replaced by any of its barycentric subdivisions we may also assume that

(c) $\operatorname{diam} g^j_{i+1}(\Delta) \leq 1/i$ for every simplex Δ in L_{i+1} and $j \leq i$ where $g^j_i = g_{j+1} \circ g_{j+2} \circ \cdots \circ g_i\colon L_i \longrightarrow L_j$.

Denote $Z = \varprojlim(L_i, g_i)$ and let $r_i\colon Z \longrightarrow L_i$ be the projections. For constructing L_{i+1} we used an arbitrary map $f\colon L_i^{[n]} \longrightarrow K(E,n), E \in \sigma(G)$. Let us show that choosing $E \in \sigma(G)$ and f in an appropriate way for each i we can achieve that $\dim_E Z \leq n$ for every $E \in \sigma(G)$ and hence $\dim_G Z \leq n$.

Let $\psi\colon F \longrightarrow K(E,n)$ be a map of a closed subset F of L_j. Then by (c) for a sufficiently large $i > j$ the map $\psi \circ g^j_i|_{(g^j_i)^{-1}(F)}$ extends over a subcomplex N of L_i to a map $\phi\colon N \longrightarrow K(E,n)$. Extending ϕ over $L_i^{[n]}$ we may assume that $L_i^{[n]} \subset N$ and replacing ϕ by its cellular approximation we assume that ϕ is cellular. Now define the map $f\colon L_i^{[n]} \longrightarrow K(E,n)$ that we use for constructing L_{i+1} as $f = \phi|_{L_i^{[n]}}$. Since g_{i+1} factors through $EW(L_i,n)$, the map $f \circ g_{i+1}|_{g^{-1}_{i+1}(L_i^{[n]})}\colon g^{-1}_{i+1}(L_i^{[n]}) \longrightarrow K(E,n)$ extends to a map $f'\colon L_{i+1} \longrightarrow K(E,n)$. Define $\psi'\colon L_{i+1} \longrightarrow K(E,n)$ by $\psi'(x) = (\phi \circ g_{i+1})(x)$ if $x \in g^{-1}_{i+1}(N)$ and $\psi'(x) = f'(x)$ otherwise. Then $\psi'|_{(g^j_{i+1})^{-1}(F)}\colon (g^j_{i+1})^{-1}(F) \longrightarrow K(E,n)$ is homotopic to $\psi \circ g^j_{i+1}|_{(g^j_{i+1})^{-1}(F)}\colon (g^j_{i+1})^{-1}(F) \longrightarrow K(E,n)$ and hence $\psi \circ g^j_{i+1}|_{(g^j_{i+1})^{-1}(F)}$ extends over L_{i+1}. Now since we need to solve only countably many extension problems for every L_j with respect to $K(E,n)$ for every $E \in \sigma(G)$ we can choose for each i a map $f\colon L_i^{[n]} \longrightarrow K(E,n)$ in the way described above to achieve that $\dim_E Z \leq n$ for every $E \in \sigma(G)$ and hence $\dim_G Z \leq n$.

The property (b) implies that for every $x \in X$ and $z \in Z$,

(d1) $g_{i+1}(\alpha_{i+1}^{-1}(st(p_{i+1}(x)))) \subset \alpha_i^{-1}(st(p_i(x)))$ and

(d2) $h_{i+1}(st((\alpha_{i+1} \circ r_{i+1})(z))) \subset st((\alpha_i \circ r_i)(z))$

where $st(a)$ =the union of all the simplexes containing a.

Define a map r: $Z \longrightarrow X$ by $r(z) = \cap\{p_i^{-1}(st((\alpha_i \circ r_i)(z))) : i = 1, 2, \ldots\}$. Then (d1) and (d2) imply that r is indeed well-defined and continuous.

The properties (d1) and (d2) also imply that for every $x \in X$

$$r^{-1}(x) = \varprojlim(\alpha_i^{-1}(st(p_i(x))), g_i|_{\alpha_i^{-1}(st(p_i(x)))}),$$

where the map $g_i|_{\alpha_i^{-1}(st(p_i(x)))}$ is considered as a map to $\alpha_{i-1}^{-1}(st(p_{i-1}(x)))$.

Since $r^{-1}(x)$ is not empty for every $x \in X$, r is a map onto and let us show that $r^{-1}(x)$ is G-acyclic.

Since $st(p_i(x))$ is contractible, $T = \alpha_i^{-1}(st(p_i(x)))$ is $(n-1)$-connected. From (a) and Proposition 2.3 it follows that $H_n(T)$ is $\mathcal{F}(G)$-torsion. Then, since G is $\mathcal{F}(G)$-torsion free, by the universal-coefficient theorem

$$H^n(T; G) = \mathrm{Hom}(H_n(T), G) = 0.$$

Thus $\tilde{\check{H}}^k(r^{-1}(x); G) = 0$ for $k \leq n$ and, since $\dim_G Z \leq n$, $\tilde{\check{H}}^k(r^{-1}(x); G) = 0$ for $k \geq n+1$. Hence r is G-acyclic and this completes the proof. ∎

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